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Second Order Linear O.D.E. and Riccati Equations*

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We prove that approximate solutions of the Riccati equation $\varphi' + \varphi^2 = a(x)$ yield asymptotic solutions $y = e^{\int^x \varphi(s) ds}$ of the second order linear equation $y'' = a(x)y$. We show that the iterative scheme $\varphi_0 = a$, $\varphi_{n+1}^2 = a - \varphi_n'$ leads to asymptotic solutions of the cited linear equation in many interesting cases.

INTRODUCTION

In this article, we shall prove that asymptotic solutions (for large x) of the second order linear ordinary differential equation

$$y'' = a(x)y \quad (0.1)$$

may be obtained from approximate solutions of the Riccati equation

$$\varphi' + \varphi^2 = a(x) \quad (0.2)$$

arising from (0.1) via the substitution $y = e^{\int^x \varphi(s) ds}$. The results to be presented here are in reality a special case of the more general results on systems of first order equations developed in a previous article [4] of the present author. However, we feel that the ease of application and great simplicity of the present results amply justifies their presentation here.

It should be remarked that most of the specific applications we present here can be obtained as applications of a theorem of Levinson [5], although Levinson's result seems considerably more difficult to apply. A special case of our results is proved by Wintner in [6]. Some of our specific applications have also been treated in previous works of the present author and John D. Dollard [1–3].

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1. SOLUTIONS OF THE LINEAR EQUATION AND ASSOCIATED RICCATI EQUATION

Our main result is the following:

(1.1) THEOREM. *Consider the equation*

$$y'' = a(x)y, \quad x_0 \leq x < \infty \quad (1.2)$$

where $a(x)$ is a real or complex valued function. Suppose that $f(x)$ and $g(x)$ are solutions of the Riccati equations:

$$f' + f^2 = a(x) + r(x); \quad g' + g^2 = a(x) + s(x) \quad (1.3)$$

where

$$f(x) \neq g(x), \quad \operatorname{Re} f(x) \geq \operatorname{Re} g(x) \quad \text{for } x_0 \leq x < \infty \quad (1.4)$$

and both $r/(f - g)$ and $s/(f - g)$ are absolutely integrable on $[x_0, \infty)$. Then:

(i) *If $f(x)$ and $g(x)$ are both real or both pure imaginary, Eq. (1.2) has two independent solutions of the form*

$$\begin{aligned} y_1(x) &= \exp \left(\int_{x_0}^x f(s) ds \right) \cdot (1 + o(1)) \\ y_2(x) &= \exp \left(\int_{x_0}^x g(s) ds \right) \cdot (1 + o(1)) \end{aligned} \quad (1.5)$$

where $o(1)$ denotes a function tending to 0 as x tends to ∞ .

(ii) *In the general case of complex f, g , (1.2) has two linearly independent solutions of the form*

$$\begin{aligned} y_1(x) &= \exp \left(\int_{x_0}^x f(s) ds \right) \cdot (1 + o(1)) \\ &\quad + c \cdot \exp \left(\int_{x_0}^x (\operatorname{Re} f(s) + i \operatorname{Im} g(s)) ds \right) \cdot (1 + o(1)) \\ y_2(x) &= \exp \left(\int_{x_0}^x g(s) ds \right) \cdot (1 + o(1)) \end{aligned} \quad (1.6)$$

where c is a constant. c may be taken to be 0 if we have

$$\int_{x_0}^{\infty} \operatorname{Re}(f - g) ds = +\infty. \quad (1.7)$$

Proof. We write (1.2) as a first order system, i.e.,

$$Y'(x) = A(x) Y(x) \quad (1.8)$$

where

$$Y(x) = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} 0 & 1 \\ a(x) & 0 \end{pmatrix}. \quad (1.9)$$

Let

$$B(x) = \begin{pmatrix} 1 & 1 \\ f(x) & g(x) \end{pmatrix} \quad (1.10)$$

where f and g are as in (1.3). Finally, let

$$Y(x) = B(x) Z(x). \quad (1.11)$$

Then $Z(x)$ satisfies the equation

$$Z'(x) = (B^{-1}AB - B^{-1}B') Z(x) \quad (1.12)$$

and we have

$$\begin{aligned} B^{-1}AB - B^{-1}B' &= \frac{1}{g-f} \begin{pmatrix} f' + fg - a & g' + g^2 - a \\ -f' - f^2 + a & -g' - fg + a \end{pmatrix} \\ &= \frac{1}{g-f} \begin{pmatrix} fg + r - f^2 & s \\ -r & -fg + g^2 - s \end{pmatrix} \\ &= \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} + \frac{1}{g-f} \begin{pmatrix} r & s \\ -r & -s \end{pmatrix} \\ &= \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} + R(x) \end{aligned} \quad (1.13)$$

where the entries of $R(x)$ are absolutely integrable on $[x_0, \infty)$. All the statements of the theorem except for the last statement are then special cases of the results of [4]. The last statement of the theorem follows immediately from the proof of the results in [4] (see (1.20), (2.10), (2.11) and the remark following (2.26) and the fact that under the hypothesis (1.7)

$$\prod_x^\infty \exp \begin{pmatrix} 0 & 0 \\ 0 & g-f \end{pmatrix} ds = \exp \int_x^\infty \begin{pmatrix} 0 & 0 \\ 0 & g-f \end{pmatrix} ds = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (1.14)$$

2. APPLICATIONS

To apply the theorem of the previous section we need to find approximate solutions to

$$\varphi + \varphi^2 = a(x). \quad (2.1)$$

A standard approach is to use an iterative method; for example, we can start with some initial function $\varphi_0(x)$ and successively determine $\varphi_n(x)$ from the equation

$$\varphi_{n+1}^2(x) = a(x) - \varphi_n'(x). \quad (2.2)$$

Starting with

$$\varphi_0(x) = 0 \quad (2.3)$$

(2.2) then yields

$$\varphi_1(x) = \sqrt{a}, \quad \varphi_2(x) = \sqrt{a - \frac{a'}{2\sqrt{a}}} \approx \sqrt{a} - \frac{a'}{4a} \quad (2.4)$$

where the approximate equality holds when a'/a is small compared with a . In order to apply Theorem (1.1) we need two approximate solutions of (2.1). These may be obtained from $\varphi_1(x)$ or $\varphi_2(x)$ of (2.4) by using the two determinations of the square root. In the case of (2.3) we may observe that $\varphi(x) = 0$ is a solution of the equation $\varphi' + \varphi^2 = 0$; another solution of this equation is given by $\varphi(x) = 1/x$. This suggests that the choice $f(x) = 1/x$, $g(x) = 0$ for the functions f, g of (1.3) may be effective in some cases. In fact we see that if $xa(x)$ is absolutely integrable for $x_0 \leq x < \infty$ then (1.3) and (1.4) are satisfied, and the equation $y'' = a(x)y$ has two independent solutions of the form $y = x \cdot (1 + o(1))$ and $y = 1 + o(1)$.

In the application of (2.4) it will be useful to write

$$a(x) = v(x) - E \quad (2.5)$$

where E is a constant so that (1.2) becomes

$$y'' = (v(x) - E)y \quad (2.6)$$

and (2.1) is

$$\varphi' + \varphi^2 = v(x) - E. \quad (2.7)$$

The function $\varphi_1(x)$ of (2.4) then becomes

$$\varphi_1(x) = \sqrt{v(x) - E}. \quad (2.8)$$

For $E \neq 0$ and $v(x)$ "small" $\phi_1(x)$ of (2.8) is approximated by $\sqrt{-E}$, and this suggests that we take f and g of Theorem (1.1) to be, respectively, $\pm \sqrt{-E}$ (where the square root is chosen so that $\sqrt{-E}$ has non-negative real part). Provided that $v(x)$ is absolutely integrable on $x_0 \leq x < \infty$, this leads to solutions of (2.6) which are of the form $y = e^{\pm \sqrt{-E} \cdot x} (1 + o(1))$. Similarly, if $E \neq 0$, $v(x) \rightarrow 0$ as $x \rightarrow \infty$, and $v'(x)$ is absolutely integrable on $x_0 \leq x < \infty$, then taking f and g of Theorem 1.1 to be $\pm \sqrt{v(x) - E}$ leads to solutions of (2.6) of the form $y = \exp(\pm \int^x \sqrt{v(s) - E} ds) \cdot (1 + o(1))$. (The behavior of the solutions of (2.6) with $E > 0$ has also been discussed in [1, 3]; this case is somewhat easier to deal with due to the boundedness of $e^{\pm \sqrt{-E} \cdot x}$.)

Certain cases not covered by the preceding examples, but in which $v(x)$ is *conditionally* integrable on $x_0 \leq x < \infty$ can be handled by some judicious guessing. For example, consider the equation

$$y'' = \left(\frac{\sin x}{x} - E \right) y, \quad E = k^2 > 0. \quad (2.9)$$

Here, both $v(x) = \sin x/x$ and v' are conditionally (but not absolutely) integrable on $x_0 \leq x < \infty$. We can in the present case show that (2.9) has two independent solutions of the form $y = e^{\pm ikx} \cdot (1 + o(1))$ *provided that* $E \neq 1/4$. Since choosing f and g of Theorem 1.1 to be $\pm ik$ fails, we try

$$f = ik + w, \quad g = -ik + z \quad (2.10)$$

with w and z to be determined. We have

$$\begin{aligned} f' + f^2 &= w' + 2ikw + w^2 - E, \\ g' + g^2 &= z' - 2ikz + z^2 - E. \end{aligned} \quad (2.11)$$

We choose w and z so that

$$w' + 2ikw = z' - 2ikz = \sin x/x. \quad (2.12)$$

Solving (2.12) yields

$$\begin{aligned} w &= e^{-2ikx} \int^x e^{2iks} \frac{\sin s}{s} ds, \\ z &= e^{2ikx} \int^x e^{-2iks} \frac{\sin s}{s} ds. \end{aligned} \quad (2.13)$$

Writing $\sin s = (1/2i)(e^{is} - e^{-is})$, an elementary integration by parts in (2.13) shows that provided $k \neq \pm(1/2)$ ($E \neq 1/4$), we can choose w and z so that both are conditionally integrable and their squares are absolutely integrable on $x_0 \leq x < \infty$. These facts together with (2.10), (2.11), (2.12)

and application of Theorem 1.1 yield the stated asymptotic form for the solutions of (2.9) (see also [1, 2] for discussion of this example from a related but different point of view).

The solutions of Eq. (2.6) when $v(x)$ becomes unbounded as $x \rightarrow \infty$ or when v tends to zero and $E = 0$ may often be analyzed by use of the approximate solution of (2.1) given by $\phi_2(x)$ (or the approximate cited expression) of (2.4). Suppose, for example, that we can pick $\sqrt{a(x)}$ with non-negative real part on $x_0 \leq x < \infty$ and then choose f and g of Theorem (1.1) to be

$$f = \sqrt{a} - a'/4a, \quad g = -\sqrt{a} - a'/4a. \quad (2.14)$$

We have

$$f' + f^2 = g' + g^2 = a + (5/16)(a'/a)^2 - (1/4)(a''/a). \quad (2.15)$$

Use of Theorem 1.1 then yields solutions of (1.2) of the form

$$\begin{aligned} y(x) &= \exp \left(\int^x \pm \sqrt{a} - \frac{a'}{4a} ds \cdot (1 + o(1)) \right) \\ &= \text{const} \cdot a^{-1/4} \exp \left(\pm \int^x \sqrt{a} ds \right) \cdot (1 + o(1)) \end{aligned}$$

in various cases when the hypotheses of that theorem are satisfied, for example, if $a(x) = \text{const} \cdot x^\alpha$ with $\alpha > -2$ or if $a(x)$ is a polynomial.

Many other cases may be analyzed similarly.

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